Normal ordering and non(anti)commutativity in open super strings

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Abstract

Nonanticommutativity in an open super string moving in the presence of a background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$ is investigated in a conformal field theoretic approach, leading to nonanticommutative structures. In contrast to several discussions, in which boundary conditions are taken as Dirac constraints, we first obtain the mode algebra by using the newly proposed normal ordering, which satisfies both equations of motion and boundary conditions. Using these the anticommutator among the fermionic string coordinates is obtained. Interestingly, in contrast to the bosonic case, this new normal ordering plays an important role in uncovering the underlying nonanticommutative structure between the fermionic string coordinates. We feel that our approach is more transparent than the previous ones and the results we obtain match with the existing results in the literature.

Keywords: Normal ordering, Boundary Conditions, Noncommutativity, Super strings

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1 Introduction

Recent progress in string theory [1, 2, 3] indicates scenarios where our four dimensional space-time with standard model fields corresponds to a D3-brane [4] embedded in a larger manifold. Now, since D-branes correspond, in type II string theories, to the space where the open string endpoints are attached, our space-time would be affected by string boundary conditions (BC). One important consequence is the possible noncommutativity of space-time coordinates at very small length scales [5, 6, 7] since commuting coordinates are incompatible with open string BC(s) in the presence of antisymmetric tensor backgrounds. This is one of the main reasons of increasing interest in several aspects of noncommutative (NC) quantum field theories [7, 8]. Furthermore, this illustrates the fact that the string BC(s) may play a crucial role in the phenomenology of four-dimensional physics.
Various methods have been applied to obtain this result [5, 9, 10, 11, 12, 13, 14, 15]. One of the most conventional methods to derive this noncommutativity is to use Dirac'procedure [16]-treat the mixed BC(s) as primary constraints [17]. However, the interpretation of the BC(s) as primary Dirac constraints lead to some ambiguities. One of them is, in contrast to the standard

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Dirac's method, one obtains infinite secondary constraint chains by consistency requirements [17, 18]. The other is that the BC(s) are valid only at the boundaries, the Dirac function is introduced so as to extend them to the neighborhood of the boundaries. In order to get a finite result, one should regularise the Dirac function. However, different regularisation procedures may lead to different results [19]; a regularisation free proof is needed.

In a series of recent papers involving bosonic strings [11] and superstrings [12], it has also been shown explicitly that noncommutativity can be obtained by modifying the canonical bracket structure, so that it is compatible with the BC(s). This is done in spirit to the treatment of Hanson et.al [20], where modified Poisson brackets (PBs) were obtained for the free Nambu-Goto string. In [13], PBs among the Fourier components were obtained using the Faddeev-Jackiw symplectic formalism [21], so that they are compatible with these BCs. Using this the PBs among the open string coordinates were computed revealing the NC structure in the string end-points. It is important to note that all these analysis were essentially confined to the classical level. In a very recent paper [14], noncommutativity in an open bosonic string moving in the presence of a background Neveu-Schwarz two-form field $B_{\mu\nu}$ is investigated in a conformal field theory approach. The mode algebra is first obtained using the newly proposed normal ordering [22], which satisfies both equations of motion and BC(s). Using these the commutator among the string coordinates is obtained. Interestingly, this new normal ordering yields the same algebra between the modes as the one satisfying only the equations of motion. In this approach, we find that noncommutativity originates more transparently and our results match with the existing results in the literature. In this paper, we shall extend the methodology of [14] to analyse an open super string propagating freely and one moving in a constant antisymmetric background field.

The organisation of this paper is as follows. In section 2, we review the recent results involving new normal ordered products (of fermionic operators) in [23]. In sec 3, we study the symplectic structure of the fermionic sector of both free and interacting super string using the method in [14] (borrowing relevant results from section 2). We conclude in section 4. The computational details of some of the key results in the paper are given in an appendix.

2 New Normal ordering for fermionic string coordinates

The action for a super string moving in the presence of a constant background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$ is given by:

$$
\begin{aligned}
S \;=\; & \frac{-1}{4\pi\alpha'}\int_{\Sigma} d\tau d\sigma \Big[\partial_a X^\mu \partial^a X_\mu + \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \\
& + i\psi_{\mu(-)}E^{\nu\mu}\partial_+\psi_{\nu(-)} + i\psi_{\mu(+)}E^{\nu\mu}\partial_-\psi_{\nu(+)} \Big]
\end{aligned} \tag{1}
$$

where, $\partial_+ = \partial_\tau + \partial_\sigma$, $\partial_- = \partial_\tau - \partial_\sigma$ and $E^{\mu\nu} = \eta^{\mu\nu} + \mathcal{B}^{\mu\nu}$.

Now since the bosonic and fermionic sectors decouple, we can consider the fermionic sector seperately[1]. The variation of the fermionic part of the action (1) gives the classical equations of motion:

$$
\partial_+\psi_{\nu(-)} = 0 \quad , \quad \partial_-\psi_{\nu(+)} = 0 \tag{2}
$$

and a boundary term that yields the following BCs:

$$
\begin{aligned}
E_{\nu\mu}\,\psi^\nu_{(+)}(0,\tau) &= E_{\mu\nu}\,\psi^\nu_{(-)}(0,\tau) \\
E_{\nu\mu}\,\psi^\nu_{(+)}(\pi,\tau) &= \lambda E_{\mu\nu}\,\psi^\nu_{(-)}(\pi,\tau)
\end{aligned} \tag{3}
$$

[1]The bosonic sector was already discussed in [14].

at the endpoints $\sigma = 0$ and $\sigma = \pi$ of the string, where $\lambda = \pm 1$ corresponds to Ramond and Neveu-Schwarz BC(s) respectively.

It is convenient now to change to complex world-sheet coordinates and therefore we first make a Wick rotation by defining $\sigma^2 = i\tau$. Then we introduce the complex world sheet coordinates [25]: $z = \sigma^1 + i\sigma^2$; $\bar{z} = \sigma^1 - i\sigma^2$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. In this notation the fermionic part of the action (1) reads:

$$S_F = \frac{-i}{4\pi\alpha'}\int_\Sigma dz d\bar{z}[\psi_{\mu(-)}E^{\nu\mu}\partial_{\bar{z}}\psi_{\nu(-)} + \psi_{\mu(+)}E^{\nu\mu}\partial_z\psi_{\nu(+)}] \tag{4}$$

while the classical equations of motion (2) and the BCs (3) take the form:

$$\partial_{\bar{z}}\psi_{\nu(-)} = 0 \ , \ \partial_z\psi_{\nu(+)} = 0 \tag{5}$$

$$\Big(E_{\nu\mu}\,\psi^\nu_{(+)}(z,\bar{z}) \,-\, E_{\mu\nu}\,\psi^\nu_{(-)}(z,\bar{z})\Big)\,|_{z=-\bar{z}\,,\,2\pi-\bar{z}} = 0\,. \tag{6}$$

We now study the properties of quantum operators corresponding to the classical variables by considering the expectation values[25]. Using the fact that the path integral of a total functional derivative vanishes and considering the insertion of one fermionic operator one finds:

$$\int [d\psi]\left[\frac{\delta}{\delta\psi^\mu_{(a)}(z,\bar{z})}[e^{-S_F}\psi^\nu_{(b)}(z',\bar{z}')]\right] = 0 \tag{7}$$

where, $a, b = +, -$. Considering first the case of $\psi^\nu_{(b)}(z',\bar{z}')$ inside the world-sheet and not at the boundary, this equation yields the following expectation values:

$$\begin{aligned}
\langle\partial_z\psi^\mu_{(+)}(z,\bar{z})\psi^\nu_{(+)}(z',\bar{z}')\rangle &= 2\,\pi\,i\,\alpha'\,\langle\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')\rangle\\
\langle\partial_{\bar{z}}\psi^\mu_{(-)}(z,\bar{z})\psi^\nu_{(-)}(z',\bar{z}')\rangle &= 2\,\pi\,i\,\alpha'\,\langle\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')\rangle\\
\langle\partial_{\bar{z}}\psi^\mu_{(-)}(z,\bar{z})\psi^\nu_{(+)}(z',\bar{z}')\rangle &= \langle\partial_z\psi^\mu_{(+)}(z,\bar{z})\psi^\nu_{(-)}(z',\bar{z}')\rangle = 0\ .
\end{aligned} \tag{8}$$

Using these results one finds the appropriate way to define normal ordered products that satisfy the equations of motion for fermionic operators that are not at the world-sheet boundary[26, 23]:

$$\begin{aligned}
:\psi^\mu_{(+)}(z,\bar{z})\ \psi^\nu_{(+)}(z',\bar{z}'): &= \psi^\mu_{(+)}(z,\bar{z})\ \psi^\nu_{(+)}(z',\bar{z}') - \frac{i\,\alpha'}{\bar{z}-\bar{z}'}\eta^{\mu\nu}\\
:\psi^\mu_{(-)}(z,\bar{z})\ \psi^\nu_{(-)}(z',\bar{z}'): &= \psi^\mu_{(-)}(z,\bar{z})\ \psi^\nu_{(-)}(z',\bar{z}') - \frac{i\,\alpha'}{z-z'}\eta^{\mu\nu}\\
:\psi^\mu_{(+)}(z,\bar{z})\ \psi^\nu_{(-)}(z',\bar{z}'): &= 0\\
:\psi^\mu_{(-)}(z,\bar{z})\ \psi^\nu_{(+)}(z',\bar{z}'): &= 0\,.
\end{aligned} \tag{9}$$

The above products satisfy the equations of motion (5) at the quantum level, but fails to satisfy the BC(s) (6).

At this point it is more convenient to choose world sheet coordinates, related to these z coordinates by conformal transformation, that simplify the representation of the boundary,

$$\omega = \exp(-iz) = e^{-i\sigma^1+\sigma^2} \ ; \ \bar{\omega} = e^{i\sigma^1+\sigma^2}. \tag{10}$$

Besides replacing $\exp(-iz) \to \omega$, we must transform the fields [26],

$$\psi^\mu_{\omega^{\frac{1}{2}}}(\omega) = (\partial_\omega z)^{\frac{1}{2}}\,\psi^\mu_{z^{\frac{1}{2}}}(z) = i^{\frac{1}{2}}\omega^{-\frac{1}{2}}\,\psi^\mu_{z^{\frac{1}{2}}}(z). \tag{11}$$

The subscripts are a reminder that these transform with half the weight of a vector. In this present coordinates the complete boundary corresponds just to the region $\omega = \bar{\omega}$. Further, the action (4) along with equations of motion (5) in terms of $\omega, \bar{\omega}$ has still the same form, while the form of BC(s) (6) change to the following:

$$\Big(E_{\nu\mu}\,\psi^{\nu}_{(+)}(\omega,\bar{\omega})\ + i\,E_{\mu\nu}\,\psi^{\nu}_{(-)}(\omega,\bar{\omega})\Big)\,|_{\omega=\bar{\omega}}\ =\ 0\,. \tag{12}$$

Let us now consider the case of an insertion of a fermionic string coordinate $\psi^{\nu}_{(\pm)}(\omega')$ located at the world-sheet boundary. Note that since $\omega' = \bar{\omega}'$ at the boundary, the fermionic coordinate insertion at the boundary depends only on ω'. Working out equation (7), but now subject to constraint (12) (with ω replaced by ω' in (12)), we find[2] (see appendix for the computational details):

$$\begin{aligned}
\langle\partial_\omega\psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega')\rangle &= 2\pi i\alpha'\langle\eta^{\mu\nu}\delta^2(\omega-\omega',\bar{\omega}-\omega')\rangle\\
\langle\partial_\omega\psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega')\rangle &= 2\pi i\alpha'\langle\eta^{\mu\nu}\delta^2(\omega-\omega',\bar{\omega}-\omega')\rangle\\
\langle\partial_\omega\psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega')\rangle &= -2\pi\alpha'\langle\Big[(\eta+\mathcal{B})^{-1}\,(\eta-\mathcal{B})\Big]^{\nu\mu}\delta^2(\omega-\omega',\bar{\omega}-\omega')\rangle\\
\langle\partial_\omega\psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega')\rangle &= 2\pi\alpha'\langle\Big[(\eta+\mathcal{B})^{-1}\,(\eta-\mathcal{B})\Big]^{\nu\mu}\delta^2(\omega-\omega',\bar{\omega}-\omega')\rangle\ .
\end{aligned} \tag{13}$$

So the appropriate normal ordering for fermionic string coordinates at the boundary reads:

$$\begin{aligned}
:\psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega'): &= \psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega')\ -\ \frac{i\alpha'}{(\bar{\omega}-\omega')}\eta^{\mu\nu}\\
:\psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega'): &= \psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega')\ -\ \frac{i\alpha'}{(\omega-\omega')}\eta^{\mu\nu}\\
:\psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega'): &= \psi^{\mu}_{(+)}(\omega,\bar{\omega})\psi^{\nu}_{(-)}(\omega')\ +\ \frac{\alpha'\Big[(\eta+\mathcal{B})^{-1}\,(\eta-\mathcal{B})\Big]^{\nu\mu}}{(\bar{\omega}-\omega')}\\
:\psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega'): &= \psi^{\mu}_{(-)}(\omega,\bar{\omega})\psi^{\nu}_{(+)}(\omega')\ -\ \frac{\alpha'\Big[(\eta+\mathcal{B})^{-1}\,(\eta-\mathcal{B})\Big]^{\nu\mu}}{(\omega-\omega')}
\end{aligned} \tag{14}$$

The above results of normal ordering of fermionic operators are new and incorporates the effect of BC(s).
Now for the functional $\mathcal{F}[X]$ (representing the combinations occuring in the left hand side of the above equation), the new normal ordering (in absence of the $\mathcal{B}$ field) can be compactly written as:

$$:\mathcal{F}:=\exp\left(\frac{i\alpha'}{2}\int d^2\omega'' d^2\omega'''\left[\frac{1}{(\omega''-\omega''')}\frac{\delta}{\delta\psi^{\mu}_{(-)}(\omega'',\bar{\omega}'')}\,\frac{\delta}{\delta\psi_{\mu(-)}(\omega''',\bar{\omega}''')}+(\omega\leftrightarrow\bar{\omega},-\leftrightarrow+)\right]\right)\mathcal{F} \tag{15}$$

Note that the fields ψ's with double prime and triple prime arguements in (15) are not located at the boundary.
We shall see now that normal ordered products are important to compute the central charge which gives us the critical dimension. The energy-momentum tensor (in the absence of the $\mathcal{B}$ field) for the fermionic sector for points inside the world-sheet (in the z-frame) is given by:

$$\begin{aligned}
T^{zz} &= -\frac{1}{2}\psi_{\mu(+)}\partial_{\bar{z}}\psi^{\mu}_{(+)} \equiv \bar{T}\\
T^{\bar{z}\bar{z}} &= -\frac{1}{2}\psi_{\mu(-)}\partial_{z}\psi^{\mu}_{(-)} \equiv T
\end{aligned} \tag{16}$$

[2]Note that the fields ψ's with unprimed arguements are not located at the boundary.

while at the boundary, the BC(s) (6) (with $\mathcal{B}=0$) relating $\psi_{\nu(-)}$ to $\psi_{\nu(+)}$ lead to:

$$\bar{T}=-\frac{1}{2}\psi_{\mu(+)}\partial_{\bar{z}}\psi^{\mu}_{(+)}=-T \tag{17}$$

where we have used $\partial_{\bar{z}}=-\partial_z$ (since $dz=-d\bar{z}$ at the boundary). The central charge can now be computed from the most singular term in the normal ordered product of energy-momentum tensor. This involves two contractions of the fermionic coordinate operator products and is proportional to [23]:

$$\begin{aligned}&\int dz'...dz''''\frac{1}{2}\left[\frac{i\alpha'}{(z'-z'')}\frac{\delta^2}{\delta\psi_{\mu(-)}(z')\delta\psi^{\mu}_{(-)}(z'')}\right]\left[\frac{i\alpha'}{(z'''-z'''')}\frac{\delta^2}{\delta\psi_{\mu(-)}(z''')\delta\psi^{\mu}_{(-)}(z'''')}\right]\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times[T(z_1)T(z_2)]\\&\sim\frac{D\alpha'^2}{4(z_1-z_2)^4}\end{aligned} \tag{18}$$

where $\sim$ mean "equal up to nonsingular terms"[3]. The above computation gives the well known result $D/2$ as the central charge where D is the dimension of space-time [26], [27]. The results are also in conformity with [23].
We shall make use of the results discussed here in the next section where we study both free and interacting open super strings.

3 Mode expansions and Non(anti)Commutativity for super strings

3.1 Free open strings

In this section, we consider the mode expansions of free ($\mathcal{B}_{\mu\nu}=0$) open super strings. We first expand $\psi^{\mu}_{(-)}(z)$ and $\psi^{\mu}_{(+)}(\bar{z})$ in Fourier modes in $(z,\bar{z})$ coordinates[26]:

$$\psi^{\mu}_{(-)}(z)=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbf{Z}}d^{\mu}_m\exp(imz)\quad;\quad\psi^{\mu}_{(+)}(\bar{z})=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbf{Z}}\tilde{d}^{\mu}_m\exp(-im\bar{z}). \tag{19}$$

Let us also write these as Laurent expansions in $(\omega,\bar{\omega})$ coordinates:

$$\psi^{\mu}_{(-)}(\omega)=\frac{i^{\frac{1}{2}}}{\sqrt{2\pi}}\sum_{m\in\mathbf{Z}}\frac{d^{\mu}_m}{\omega^{m+\frac{1}{2}}}\quad;\quad\psi^{\mu}_{(+)}(\bar{\omega})=\frac{i^{-\frac{1}{2}}}{\sqrt{2\pi}}\sum_{m\in\mathbf{Z}}\frac{\tilde{d}^{\mu}_m}{\bar{\omega}^{m+\frac{1}{2}}}. \tag{20}$$

Now the BC(s) (12) in case of free open super strings ($\mathcal{B}_{\mu\nu}=0$) requires $d=\tilde{d}$ in the expansions (20). The expressions (20) can be equivalently written as:

$$d^{\mu}_m\;=\;\frac{\sqrt{2\pi}}{\sqrt{i}}\oint\frac{d\omega}{2\pi i}\,\omega^{m-\frac{1}{2}}\,\psi^{\mu}_{(-)}(\omega)=-\sqrt{2\pi}\sqrt{i}\oint\frac{d\bar{\omega}}{2\pi i}\,\bar{\omega}^{m-\frac{1}{2}}\,\psi^{\mu}_{(+)}(\bar{\omega}). \tag{21}$$

The anticommutation relation between d's can be worked out from the contour arguement [25] and the operator product expansion (OPE) (14) (with $\mathcal{B}_{\mu\nu}=0$):

$$\begin{aligned}\{d^{\mu}_m,d^{\nu}_n\}&=\frac{1}{i}\oint\frac{d\omega_2}{2\pi i}\mathrm{Res}_{\omega_1\to\omega_2}\left(\omega_1^{m-\frac{1}{2}}\,\psi^{\mu}_{(-)}(\omega_1)\,\omega_2^{n-\frac{1}{2}}\,\psi^{\nu}_{(-)}(\omega_2)\right)\\&=2\pi\alpha'\,\eta^{\mu\nu}\,\delta_{m+n,0}=\eta^{\mu\nu}\,\delta_{m+n,0}\end{aligned} \tag{22}$$

[3]The other less singular terms are not given explicitly.

where we have set $2\pi\alpha' = 1$. The anti-commutation relations between $\psi^{\mu}_{(-)}(\omega,\bar{\omega})$ and $\psi^{\nu}_{(+)}(\omega',\bar{\omega'})$ are then obtained by using (22):

$$\begin{aligned}
\left\{\psi^{\mu}_{(-)}(\omega,\bar{\omega}),\psi^{\nu}_{(-)}(\omega',\bar{\omega'})\right\} &= \frac{i\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left(\omega^{-m-\frac{1}{2}}\,\omega'^{m-\frac{1}{2}}\right) \\
\left\{\psi^{\mu}_{(+)}(\omega,\bar{\omega}),\psi^{\nu}_{(+)}(\omega',\bar{\omega'})\right\} &= -\frac{i\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left(\bar{\omega}^{-m-\frac{1}{2}}\,\bar{\omega}'^{m-\frac{1}{2}}\right) \\
\left\{\psi^{\mu}_{(-)}(\omega,\bar{\omega}),\psi^{\nu}_{(+)}(\omega',\bar{\omega'})\right\} &= \frac{\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left(\omega^{-m-\frac{1}{2}}\,\bar{\omega}'^{m-\frac{1}{2}}\right). \qquad (23)
\end{aligned}$$

To obtain the usual equal time ($\tau = \tau'$) anticommutation relation we first rewrite (23) in "z frame" using (10, 11) and then in terms of σ^1, σ^2 to find:

$$\begin{aligned}
\left\{\psi^{\mu}_{(-)}(\sigma^1,\sigma^2),\psi^{\nu}_{(-)}(\sigma^{1\prime},\sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left[\exp\left(im(\sigma^1 + i\sigma^2 - \sigma'^1 - i\sigma'^2)\right)\right] \\
\left\{\psi^{\mu}_{(+)}(\sigma^1,\sigma^2),\psi^{\nu}_{(+)}(\sigma^{1\prime},\sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left[\exp\left(im(\sigma^1 - i\sigma^2 - \sigma'^1 + i\sigma'^2)\right)\right] \\
\left\{\psi^{\mu}_{(-)}(\sigma^1,\sigma^2),\psi^{\nu}_{(+)}(\sigma^{1\prime},\sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi}\sum_{m\in\mathbf{Z}}\left[\exp\left(im(\sigma^1 + i\sigma^2 + \sigma'^1 - i\sigma'^2)\right)\right]. \qquad (24)
\end{aligned}$$

Finally substituting $\tau = \tau'$ (i.e. $\sigma^2 = \sigma^{2\prime}$) and $\sigma^1 = \sigma$ we get back the equal time anti-commutation relations:

$$\begin{aligned}
\left\{\psi^{\mu}_{(-)}(\sigma,\tau),\psi^{\nu}_{(-)}(\sigma',\tau)\right\} &= \eta^{\mu\nu}\delta_P(\sigma-\sigma') \\
\left\{\psi^{\mu}_{(+)}(\sigma,\tau),\psi^{\nu}_{(+)}(\sigma',\tau)\right\} &= \eta^{\mu\nu}\delta_P(\sigma-\sigma') \\
\left\{\psi^{\mu}_{(-)}(\sigma,\tau),\psi^{\nu}_{(+)}(\sigma',\tau)\right\} &= \eta^{\mu\nu}\delta_P(\sigma+\sigma'). \qquad (25)
\end{aligned}$$

where, $\delta_P(\sigma-\sigma')$ is the so called periodic delta function which is defined as:

$$\delta_P(\sigma-\sigma') = \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\exp\left(im(\sigma-\sigma')\right). \qquad (26)$$

This structure of anticommutator is completely consistent with the BCs (12) for $\mathcal{B}_{\mu\nu} = 0$. Note that not only the usual Dirac delta function is replaced by the periodic delta function but also the anticommutator among $\psi_{(-)}, \psi_{(+)}$ are non-vanishing even in case of the free open fermionic string [12, 13]. The most important feature of the above analysis is that unlike the bosonic case, the new normal ordering of the fermionic operators (that incorporates the BC(s)) (14) leads to the nonanticommutative structures (25) among the fermionic string coordinates.

3.2 Open superstring in the constant $\mathcal{B}$-field background

We now analyse the open superstring moving in presence of a background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$. To begin with, let us again consider the Laurent expansion of $\psi^{\mu}_{(-)}(\omega)$ and $\psi^{\mu}_{(+)}(\bar{\omega})$ (20). Now due to the BC(s) (12) (with $\mathcal{B}_{\mu\nu} \neq 0$), the modes d and $\tilde{d}$ are no longer independent but satisfy the following relation:

$$E_{\mu\nu}\,d^{\nu}_m = E_{\nu\mu}\,\tilde{d}^{\nu}_m. \qquad (27)$$

Hence there exists only one set of independent modes α^μ_m, which can be thought of as the modes of free open strings and is related to d^μ_m and $\tilde{d}^\mu_m$ by:

$$\begin{aligned} d^\mu_m &= (\delta^\mu{}_\nu - \mathcal{B}^\mu{}_\nu)\,\alpha^\nu_m := [(1\!\!1-\mathcal{B})\alpha]^\mu_m \\ \tilde{d}^\mu_m &= (\delta^\mu{}_\nu + \mathcal{B}^\mu{}_\nu)\,\alpha^\nu_m := [(1\!\!1+\mathcal{B})\alpha]^\mu_m\,. \end{aligned} \tag{28}$$

Note that under world-sheet parity transformation (i.e. $\sigma \leftrightarrow -\sigma$), $d^\mu_m \leftrightarrow \tilde{d}^\mu_m$, since $\mathcal{B}_{\mu\nu}$ is a world-sheet pseudo-scalar (similar to bosonic part [14]). Substituting (28) in (20), we obtain the following Laurent expansions for ψ^μ_- and ψ^μ_+:

$$\psi^\mu_{(-)}(\omega) = \frac{i^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m\in\mathbf{Z}} \frac{[(1\!\!1-\mathcal{B})\alpha]^\mu_m}{\omega^{m+\frac{1}{2}}} \tag{29}$$

$$\psi^\mu_{(+)}(\bar{\omega}) = \frac{i^{-\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m\in\mathbf{Z}} \frac{[(1\!\!1+\mathcal{B})\alpha]^\mu_m}{\bar{\omega}^{m+\frac{1}{2}}}.$$

These are the appropriate mode expansions for the fermionic part of the interacting superstring, that satisfy both the equations of motion (5) and the BC(s) (12).
Now the expressions (29) for interacting superstrings can also be written as:

$$\begin{aligned} [(1\!\!1-\mathcal{B})\alpha]^\mu_m &= \frac{\sqrt{2\pi}}{i} \oint \frac{d\omega}{2\pi i}\,\omega^{m-\frac{1}{2}}\,\psi^\mu_{(-)}(\omega) \\ [(1\!\!1+\mathcal{B})\alpha]^\mu_m &= \frac{\sqrt{2\pi}}{i} \oint \frac{d\bar{\omega}}{2\pi i}\,\bar{\omega}^{m-\frac{1}{2}}\,\psi^\mu_{(+)}(\bar{\omega}). \end{aligned} \tag{30}$$

The anticommutation relation between α's can be obtained once again from the contour arguement (using (30)) and the $\psi\,\psi$ OPE (14):

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \left[\left(1\!\!1-\mathcal{B}^2\right)^{-1}\right]^{\mu\nu} \delta_{m,-n} = \left(\mathcal{M}^{-1}\right)^{\mu\nu} \delta_{m,-n} \tag{31}$$

where, $\mathcal{M} = (1\!\!1 - \mathcal{B}^2)$; $(\mathcal{B}^2)^{\mu\nu} = \mathcal{B}^\mu{}_\rho \mathcal{B}^{\rho\nu}$[4]. Now the anticommutator between the fermionic string coordinates can be computed using (29), (31). The antibrackets between $\left\{\psi^\mu_{(-)}, \psi^\nu_{(-)}\right\}$ and $\left\{\psi^\mu_{(+)}, \psi^\nu_{(+)}\right\}$ are the same as that of free case but the anticommutator between $\psi^\mu_{(-)}$ and $\psi^\mu_{(+)}$ gets modified to the following form:

$$\{\psi^\mu_-(\omega,\bar{\omega}), \psi^\nu_+(\omega',\bar{\omega}')\} = \frac{1}{2\pi} \sum_{m\in\mathbf{Z}} \left[\frac{(1\!\!1-\mathcal{B})^\mu{}_\rho \left[\left(1\!\!1-\mathcal{B}^2\right)^{-1}\right]^{\rho\sigma} (1\!\!1-\mathcal{B})_\sigma{}^\nu}{\omega^{m+\frac{1}{2}}\,\bar{\omega}'^{-m+\frac{1}{2}}}\right]. \tag{32}$$

Now proceeding as before, we can write the above anticommutation relation in (τ,σ) coordinates to obtain the usual equal time (i.e. $\tau = \tau'$) anticommutation relation:

$$\left\{\psi^\mu_{(-)}(\sigma,\tau), \psi^\nu_{(+)}(\sigma',\tau)\right\} = E^{\rho\,\mu} \left[\left(1\!\!1-\mathcal{B}^2\right)^{-1}\right]_{\rho\,\sigma} E^{\nu\,\sigma}\,\delta_P(\sigma+\sigma'). \tag{33}$$

The above result reduces to the the free case result in the $\mathcal{B}_{\mu\nu} = 0$ limit and also agrees with the existing results in the literature [12], [13].

[4]Here we should note that $(1\!\!1)^{\mu\nu} = \eta^{\mu\nu}$.

4 Conclusions

In this paper, we have used conformal field theoretic techniques to compute the anticommutator among Fourier components of fermionic sector of super strings. Using this the anticommutator between the basic fermionic fields is obtained. This is the extension of our earlier work on bosonic strings [14]. The method is also different from ([13]), where the algebra among the Fourier components have been computed using the Faddeev-Jackiw symplectic formalism. The advantage of this approach (as mentioned in [14] also) is that the results one obtains takes into account the quantum effects right from the beginning, in contrary to the previous investigations, which were made essentially at the classical level [5, 9, 11, 13, 24]. Interestingly, the new normal ordering that takes into account the effect of the BC(s) plays a crucial role in obtaining the nonanticommutative symplectic structure among the fermionic string coordinates (25). This is in contrast to the analysis in case of the bosonic strings where the new normal ordering has no bearing on the symplectic structure. Finally, we also computed the oscillator algebra in presence of the $\mathcal{B}$ field which is a parity-odd field on the string world-sheet. As in the bosonic case, in presence of this $\mathcal{B}$ field, the fourier modes appearing in the Laurent series expansions of the fermionic fields $\psi^\mu_{(-)}$ and $\psi^\mu_{(+)}$ of the closed string are no longer equal when open string BCs (12) are imposed. These rather get related to the free oscillator modes d^μ_m. Using these expressions of the modes (28), we rewrite the fermionic fields $\psi^\mu_{(-)}$ and $\psi^\mu_{(+)}$ entirely in terms of the free oscillator modes α^μ_m (28). Then a straight forward calculation, involving $\psi\psi$ OPE (14) and contour arguement yields the NC anticommutator given in (33), thereby reproducing the results of [12].

Appendix

Here we would like to give some of the computational details involved in deriving (13) from (7) and (12) (for convenience we treat the free case, i.e. $\mathcal{B} = 0$). Eq.(7) with z replaced by ω yields:

$$\begin{aligned} 0 &= \int [d\psi] \left[\frac{\delta}{\delta \psi^\mu_{(a)}(\omega, \bar{\omega})} [e^{-S_F} \psi^\nu_{(b)}(\omega', \bar{\omega}')] \right] \\ &= \int [d\psi] e^{-S_F} \left[-\frac{\delta S_F}{\delta \psi_{(a)}(\omega, \bar{\omega})} \psi_{(b)}(\omega', \bar{\omega}') + \frac{\delta \psi_{(b)}(\omega', \bar{\omega}')}{\delta \psi_{(a)}(\omega, \bar{\omega})} \right] \end{aligned} \tag{34}$$

Putting $a = +$, $b = -$; we obtain:

$$0 = \int [d\psi] e^{-S_F} \left[\frac{i}{2\pi\alpha'} \partial_\omega \psi_+(\omega, \bar{\omega}) \psi_-(\omega', \bar{\omega}') + \frac{\delta \psi_{(-)}(\omega', \bar{\omega}')}{\delta \psi_{(+)}(\omega, \bar{\omega})} \right] \tag{35}$$

where we have dropped the boundary term (arising from the first term in (34)) as it vanishes due to the BC(s) (12) (with $\mathcal{B} = 0$).

Now we discuss two distinct cases seperately.

• Case 1: The insertion $\psi_{(-)}(\omega', \bar{\omega}')$ is not located at the boundary:
In this case

$$\frac{\delta \psi_{(-)}(\omega', \bar{\omega}')}{\delta \psi_{(+)}(\omega, \bar{\omega})} = 0 \tag{36}$$

and therefore one finds:

$$\langle \partial_\omega \psi^\mu_{(+)}(\omega, \bar{\omega}) \psi^\nu_{(-)}(\omega', \bar{\omega}') \rangle = 0. \tag{37}$$

• Case 2: The insertion $\psi_{(-)}(\omega')$ is located at the boundary (since $\omega' = \bar{\omega}'$ at the boundary, the insertion $\psi_{(-)}(\omega')$ depends only on the arguement ω'):
In this case the computation of the second term in (35) needs to be done more carefully. One finds

$$\begin{aligned} \frac{\delta\psi_{(-)}(\omega',\bar{\omega}')}{\delta\psi_{(+)}(\omega,\bar{\omega})}\Big|_{\omega'=\bar{\omega}'} &= i\,\frac{\delta\psi_{(+)}(\omega',\bar{\omega}')}{\delta\psi_{(+)}(\omega,\bar{\omega})}\Big|_{\omega'=\bar{\omega}'} \\ &= i\,\delta^2\left(\omega-\omega',\bar{\omega}-\bar{\omega}'\right)\Big|_{\omega'=\bar{\omega}'} \\ &= i\,\delta^2\left(\omega-\omega',\bar{\omega}-\omega'\right). \end{aligned} \tag{38}$$

where we have used the BC (12) (with ω repaced by ω') in the first line of (38).
Substituting (38) in (35) and equating the volume term to zero, one finds the third of the equations in (13) (with $\mathcal{B} = 0$).
Similarly, for other choices of a, b the rest of the equations in (13) can be derived (with $\mathcal{B} = 0$).

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References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.

[2] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

[3] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

[4] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[5] C.S. Chu, P.M. Ho, Nucl.Phys.B 550 (1999) 151.

[6] A.Y. Alekseev, A. Recknagel, V. Schomerus, JHEP 9909 (1999) 023.

[7] N. Seiberg, E. Witten, JHEP 9909 (1999) 032.

[8] R.J. Szabo, Phys.Rep. 378 (2003) 207.

[9] C.S. Chu, P.M. Ho, Nucl.Phys.B 568 (2000) 447.

[10] F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari, Nucl.Phys.B 576 (2000) 578.

[11] R. Banerjee, B. Chakraborty, S. Ghosh, Phys. Lett. B 537, 340 (2002).

[12] B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra, F. G. Scholtz, Phys. Letts. **B 625** (2005), 302, [hep-th/0508156].

[13] Jian Jing, Zheng-Wen Long, Phys. Rev **D 72**, 126002 (2005); Jian Jing, Phys. Rev **D 73**, 086001 (2006).

[14] B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra, Phys. Rev. D 74, 105011 (2006), [hep-th/0608165].

[15] S. Gangopadhyay, A. Ghosh Hazra, A. Saha, Phys. Rev. D 74, 125023 (2006), [hep-th/0701012].

[16] P. A. M. Dirac, *Lectures on Quantum Mechanics*, (Yeshiva University Press, New York, 1964).

[17] M.M. Sheikh Jabbari and A. Shirzad, Eur. Phys. J. C 19, 383 (2001).

[18] M. Dehghani and A. Shirzad, [hep-th/ 0511248].

[19] F. Loran, Phys. Lett. B 544, 199 (2002).

[20] A.J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian System*, Roma, Accademia Nazionale Dei Lincei, (1976).

[21] L. D. Faddeev and R. Jackiw, Phys. Rev. Lett **60** (1988), 1692.

[22] N. R. F. Braga, H. L. Carrion, C. F. L. Godinho, J. Math. Phys 46, 062302 (2005), [hep-th/0412075].

[23] N. R. F. Braga, H. L. Carrion, C. F. L. Godinho, Phys. Letts. **B 638** (2006), 272, [hep-th/ 0602212].

[24] N. R. F. Braga, C. F. L. Godinho, [hep-th/ 0110297] Phys. Rev. D 65 (2002) 085030.

[25] J. Polchinski, *String Theory*, Vol. I, Cambridge University Press, 1998.

[26] J. Polchinski, *String Theory*, Vol. II, Cambridge University Press, 1998.

[27] M. Kaku, Strings, Conformal Fields, and M- Theory, Springer-Verlag New York, 2000 (second edition).